PROCEEDINGS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On a Property Which Holds Good for all Groupings of a Normal Distribution of Frequency for Two Variables, with Applications to the Study of Contingency-Tables for the Inheritance of Unmeasured Qualities

G. Udny Yule

Proc. R. Soc. Lond. A 1906 **77**, 324-336
doi: 10.1098/rspa.1906.0029

Email alerting service Receive free email alerts when new articles cite this article - sign up in the box at the top right-hand corner of the article or click **here**

To subscribe to *Proc. R. Soc. Lond. A* go to: **http://rspa.royalsocietypublishing.org/subscriptions**

This journal is © 1906 The Royal Society

On a Property which holds good for all Groupings of a Normal Distribution of Frequency for Two Variables, with Applications to the Study of Contingency-Tables for the Inheritance of Unmeasured Qualities.

By G. UDNY YULE, Newmarch Lecturer in Statistics, University College, London.

(Communicated by Professor O. Henrici, F.R.S. Received November 4,—Read December 7, 1905.)

I. *General Theory and Illustrations from Correlation Tables.*

1. Suppose a contingency-table to have been formed for two characters A and B which have been assigned in some way into classes $A_1 \ldots A_r$, $B_1 \ldots B_s$, the table having r columns, s rows, and giving the frequencies of occurrence of all combinations like A_mB_n. Let (A_m) (B_n) denote the frequencies of A_m and B_n, (A_mB_n) the frequency of their combination.

Extract from the general contingency-table any four adjacent frequencies, say—

$$\begin{array}{ll} (A_mB_n), & (A_{m+1}B_n) \\ (A_mB_{n+1}), & (A_{m+1}B_{n+1}). \end{array}$$

Consider these as the frequencies of a table exhibiting the association between A_m and B_n in a "universe" or field of observation limited to A_m A_{m+1}, B_n B_{n+1} alone. If the ratio

$$(A_mB_n)(A_{m+1}B_{n+1})/(A_mB_{n+1})(A_{m+1}B_n)$$

be unity, A_m and B_n are independent; if greater than unity, positively associated; if less than unity negatively associated.

The whole of the contingency-table can be analysed into a series of elementary tetrads such as the above, each one overlapping its neighbours, so that an r-s-fold table contains $(r-1)(s-1)$ tetrads. Suppose the sign of the association in each tetrad to be noted. These signs may (1) vary significantly in different parts of the table, or (2) be the same throughout.

A distribution of the last kind possesses several interesting and useful properties. To adopt a word from the writers on elasticity, it may be termed an *isotropic* distribution.

2. In an isotropic distribution the sign of the association is the same not only for every elementary tetrad of adjacent frequencies, but for every set of four frequencies in the compartments common to two rows and two columns, *e.g.* $(A_m B_n)$, $(A_{m+p} B_n)$, $(A_m B_{n+q})$, $(A_{m+p} B_{n+q})$.

The Royal Society is collaborating with JSTOR to digitize, preserve, and extend access to
Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character. www.jstor.org

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PROCEEDINGS OF THE ROYAL SOCIETY A

Suppose that the sign of association in the elementary tetrads is positive, so that—

$$(A_m B_n)\,(A_{m+1} B_{n+1}) > (A_{m+1} B_n)\,(A_m B_{n+1}), \qquad (1)$$

and similarly,

$$(A_{m+1} B_n)\,(A_{m+2} B_{n+1}) > (A_{m+2} B_n)\,(A_{m+1} B_{n+1}). \qquad (2)$$

Then, multiplying up and cancelling, we have

$$(A_m B_n)\,(A_{m+2} B_{n+1}) > (A_{m+2} B_n)\,(A_m B_{n+1}). \qquad (3)$$

That is to say, the association is still positive though the two A-arrays are not adjacent.

3. An isotropic table remains isotropic in whatever way it may be condensed by grouping together adjacent rows or columns.

Thus, from (1) and (3) we have, adding,

$$(A_m B_n)\,[(A_{m+1} B_{n+1}) + (A_{m+2} B_{n+1})] > (A_m B_{n+1})\,[(A_{m+1} B_n) + (A_{m+2} B_n)]. \qquad (4)$$

That is to say, the sign of the elementary association is unaffected by throwing the $(m+1)$th and $(m+2)$th A-arrays into one.

4. As the extreme case of the preceding theorem, we may suppose both rows and columns grouped and regrouped until only a 2×2-fold table is left; we then have the theorem :—

If an isotropic distribution be reduced to a fourfold distribution in any way whatever, the sign of the association in such fourfold table is the same as in the elementary tetrads of the original table.

5. Isotropy, therefore, is a quality that cannot be destroyed by any mode of grouping, or of extraction, of arrays. If the smallness of the number of observations in any array (in a practical case) render the discussion of approximate isotropy difficult, owing to the influence of "errors of sampling," we may either drop that array and treat those on either side of it as adjacent, or we may group it with one or more adjacent arrays. The latter process may of course conceal, but it cannot create, a departure from isotropy. As a matter of practice no correlation-table with ordinary fineness of grouping could be expected to exhibit strict isotropy; the elementary associations are too small and the probable errors too large. Some grouping is essential—as an extreme case grouping to 3×3-fold form. In such a reduced table there will be only four associations to inspect in order to determine the isotropy, viz., those corresponding to tetrads 1254, 2365, 5698, 4587 in the 1st, 2nd, 3rd and 4th quadrants of the table, taking the compartments as numbered in the order

1	2	3
4	5	6
7	8	9

Both grouping and extraction of arrays have been freely used in Part II of this paper for the discussion of approximate isotropy.

6. The normal frequency-distribution for two variables is an isotropic distribution, to which the preceding theorems accordingly apply.

Let the frequency for an interval of $\pm\frac{1}{2}\text{-}dx \pm \frac{1}{2}\text{-}dy$ round deviations of the variables x, y be given by $z\,.\,dx\,.\,dy$, where

$$z = z_0 \exp - \tfrac{1}{2}\left(\frac{x^2}{c_1{}^2}+\frac{y^2}{c_2{}^2}-\frac{2rxy}{c_1c_2}\right), \tag{5}$$

c_1c_2 being the standard-deviations of arrays and r the coefficient of correlation. Writing down from (5) the frequencies of pairs of deviations, x_1y_1, x_1y_2, x_2y_1, x_2y_2, we have for the cross-ratio of the frequencies (x_1y_1), etc.,

$$\frac{(x_1y_1)\,(x_2y_2)}{(x_1y_2)\,(x_2y_1)} = \exp\frac{r}{c_1c_2}(x_2-x_1)\,(y_2-y_1). \tag{6}$$

Assuming that x_2-x_1 is of the same sign as y_2-y_1, the index on the right is of the same sign as r. The sign of the association in the elementary tetrad is therefore of the same sign as r. The distribution is therefore isotropic. Every grouping of a normal distribution is therefore isotropic—be the class-intervals equal or unequal, large or small—and the sign of the association in a normal distribution reduced to 2 × 2-fold form is always the same, whatever the axes of division.*

7. In Professor Pearson's earlier method of calculating the coefficient of correlation for unmeasured qualities, it is assumed that the contingency-table represents a grouping, by arbitrary intervals, of normally distributed frequency.† But the assumption of normality can only be valid for some one order of the classes—at most—and the question arises whether the right order has been assigned; Professor Pearson's later method of the "contingency-coefficient" avoids all difficulty as to class-order, and this is rightly claimed as one of its chief advantages.‡

If, however, the contingency-tables dealt with were really disarrangements of normally distributed frequency, no great difficulty as to class-orders should arise. If the rows and columns of an isotropic distribution be disarranged in any way whatever, they can be rearranged in their right order, by inspection almost, owing to the following property:—

If, say, two columns in an isotropic contingency-table be isolated, the

---

* In a previous memoir (*vide* bibliography, p. 336, 15, 1900) I showed that this was empirically the case for a certain approximately normal distribution, but did not recognise it as a necessary consequence of normality.

† 'Bibl.,' 5, 1901; the method employed in numerous memoirs of later date (1, 2, 3, 4, 6, 7, 9, 10).

‡ 'Bibl.,' 11, 1904.

PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

ratios of the frequency in the one column to the sum of the frequencies in the two columns, for successive rows, must form a continuous series, ascending or descending.

8. Thus, suppose we isolate the mth and $(m+p)$th A-arrays with frequencies

$$\begin{array}{ll} (A_m B_1), & (A_{m+p} B_1), \\ (A_m B_2), & (A_{m+p} B_2), \\ \vdots & \vdots \\ (A_m B_s), & (A_{m+p} B_s), \end{array}$$

we have from the first tetrad, at the head of the column, supposing the fundamental association positive,

$$\frac{(A_m B_1)}{(A_m B_1)+(A_{m+p} B_1)} > \frac{(A_m B_2)}{(A_m B_2)+(A_{m+p} B_2)}.$$

From the second tetrad

$$\frac{(A_m B_2)}{(A_m B_2)+(A_{m+p} B_2)} > \frac{(A_m B_3)}{(A_m B_3)+(A_{m+p} B_3)},$$

and so on, each ratio being less than the one above it.

If the rows have been disarranged, their right order will be given at once by the order of the ratios (subject, as always in practical cases, to difficulties created by casual irregularities of the nature of errors of sampling or otherwise).

It is assumed, of course, that the orders 1, 2, 3 ... r, r ... 3, 2, 1 are indifferent; it is, *e.g.*, mere convention whether we agree to range a series of eye-colours in order from light to dark or from dark to light.

9. This possibility of assigning the right order of arrays from the intrinsic character of the distribution, without reference to extraneous considerations, seemed so extremely curious that I judged it worth while to work out an illustration in a case where the right order was known *à priori*, so as to provide a check. The figures used are given in Table I (p. 328).

The two columns with the greatest total frequencies are 4 and 6, and these are best chosen for a first comparison as being least affected by "errors of sampling." Working out the successive ratios 53·25/(53·25+86·5) 75·75/(75·75+50), etc., we have the series of ratios (1) below. This gives as the order for the B's:—(9, 10), 3, 5, 2, 4, 1, 6, 7, 8, the correct order *inter se* of 9 and 10 being uncertain. Checking this result by a similar use of columns 6 and 9 we get the series (2), which gives the same order reversed in sign. Either A_4 and A_6 or A_6 and A_9 must therefore have been disarranged. Assuming that A_6A_9 are right and that the fundamental association is positive, we may take the order as 8, 7, 6, 1, 4, 2, 5, 3, (9, 10).

PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

As regards the last pair, a comparison of columns 4 and 8 or 4 and 9 suggests the order 9, 10 whilst a comparison of 9 with 8 (this pair being disarranged) suggests 10, 9. The preference lies slightly with the former order.

Table I.—Anisotropic Contingency-Table for Two Qualities, A and B (Table II giving the Isotropic Rearrangement).

Quality A.

Quality B.	1	2	3	4	5	6	7	8	9	Total.
1	53	15·75	5	53·25	—	86·5	—	2·5	21·5	237·5
2	16·5	1	—	75·75	—	50	4·5	22·75	65·5	236
3	1·5	1	—	10·5	—	1·5	2·5	9	11·5	37·5
4	38	14·75	2·5	106	—	90	1	17·5	53·25	323
5	4	—	—	32·5	—	11·5	1·5	20·75	34·75	105
6	35·25	12	3	11·75	3	30·5	—	—	4·5	100
7	7·25	5	1	1·75	—	8·5	—	—	0·5	24
8	1·5	1	—	—	—	1	—	—	—	3·5
9	—	—	—	3	—	—	—	3·5	1·5	8
10	—	—	—	1	—	—	—	1·5	1	3·5
Total	157	50·5	11·5	295·5	3	279·5	9·5	77·5	194	1078

Ratio $(A_mB_n)/\{(A_mB_n)+(A_{m+p}B_n)\}$.

Row.	(1) $m = 4, m+p = 6$.	(2) $m = 6, m+p = 9$.
1	0·331	0·801
2	0·602	0·433
3	0·875	0·115
4	0·541	0·628
5	0·739	0·249
6	0·278	0·872
7	0·171	0·945
8	0	1
9	1	0
10	1	0

10. Next proceeding to the quality A and dealing with rows precisely as we have done with columns, we find the following ratios for the pairs of rows 1, 2 and 2, 4:—

Ratio $(A_mB_n)/\{(A_mB_n)+(A_mB_{n+q})\}$.

Column.	(1) $n = 1, n+q = 2$.	(2) $n = 2, n+q = 4$.
1	0·763	0·303
2	0·941	0·064
3	1	0
4	0·413	0·417
5	?	?
6	0·634	0·357
7	0	0·818
8	0·099	0·565
9	0·247	0·552

Here the position of 5 is indeterminate, but remembering that B_2 and B_4 are disarranged, both series concur in giving the order for the remainder: 3, 2, 1, 6, 4, 9, 8, 7. Our general knowledge of the attributes would certainly be called into play in any practically treated case in assigning the position to A_5; the condition of isotropy alone can only offer a slight suggestion that it should lie near the upper end of the series, seeing that the combination A_5B_6 is the only one occurring and that B_6 lies near the upper end. The order may then be approximately 5 (?), 3, 2, 1, 6, 4, 9, 8, 7.

11. If rows and columns be now rearranged in the orders determined we get the comparatively orderly-looking Table II, which is, as a matter of fact,

Table II.—Showing Table I rearranged in (approximately) Isotropic Order. The numbers at heads of columns and rows refer to Table I.

Quality A.

Quality B.		5	3	2	1	6	4	9	8	7	Total.
	8	—	—	1	1·5	1	—	—	—	—	3·5
	7	—	1	5	7·25	8·5	1·75	0·5	—	—	24
	6	3	3	12	35·25	30·5	11·75	4·5	—	—	100
	1	—	5	15·75	53	86·5	53·25	21·5	2·5	—	237·5
	4	—	2·5	14·75	38	90	106	53·25	17·5	1	323
	2	—	—	1	16·5	50	75·75	65·5	22·75	4·5	236
	5	—	—	—	4	11·5	32·5	34·75	20·75	1·5	105
	3	—	—	1	1·5	1·5	10·5	11·5	9	2·5	37·5
	9	—	—	—	—	—	3	1·5	3·5	—	8
	10	—	—	—	—	—	1	1	1·5	—	3·5
	Total	3	11·5	50·5	157	279·5	295·5	194	77·5	9·5	1078

only a grouping by 2-inch intervals of a table of Professor Pearson's for inheritance of stature from father to son (8, 1903, Table XXII). The columns should be headed "Stature of Father," "57″·5—59″·5," "59″·5—61″·5," etc. (commencing at the left), the rows "Stature of Son," "59″·5—61″·5," "61″·5—63″·5," etc. (commencing at the top). Table I was a disarrangement of Table II, deliberately effected for purposes of experiment.

12. The success of the experiment suggests that while errors will very likely be made as regards the placing of infrequent classes like A_5, the method should be able to assign positions to the more frequent classes with a considerable degree of certainty, provided always that the assumption of isotropy is legitimate.

For tables based on anthropometric measurements I think this assumption will in general hold. I tested 14 of Professor Pearson's tables for stature, span, fore-arm, and head measurements by the rough process of grouping to 3 × 3-fold form (§ 5), using in some cases two different groupings. The result was the same in every case, the grouped tables being isotropic without exception. The class-limits were so chosen as not to make any frequencies very small, but otherwise as much variety as possible was admitted.[*] It is probable enough that more extensive trials on similar material will yield some anisotropic tables, but seems unlikely that significant anisotropy will prove anything but exceptional.

II. *Applications of the Preceding Principles to Published Contingency-Tables for the Inheritance of Unmeasured Characters.*

13. The first contingency-tables to which I endeavoured to apply the principles of the preceding sections were the tables for inheritance of eye-colours, based on Mr. Galton's data, which are given in Professor Pearson's Memoir (6), 1900. In these tables the colours are classified under eight heads:—(1) Light blue; (2) blue, dark blue; (3) blue green, grey; (4) dark grey, hazel; (5) light brown; (6) brown; (7) dark brown; (8) very dark brown, black. The tints, 1, 5, and 8 are relatively infrequent, at least in the tables for direct parental inheritance; as a first rough trial I therefore dropped these tints and endeavoured by the methods of §§ 8 to 11 to determine the true order *inter se* of the remaining tints 2, 3, 4, 6, 7, using

[*] The following are the tables cited from Pearson (8) and (9), with the groupings used as given by the limits to the central class; where two are given, two distinct groupings were tested. From (8):—XXII, stature, 65·5—69·5 and 67·5—69·5; XXIII, span, 69—71; XXIV, fore-arm, 18—19; XXXI, stature, 61—63 and 62—64; XXXII, span, 62—64; XXXIII fore-arm, 16—17. From (9):—E_1, cephalic index, 78—80 and 78—82; E_2, ditto, 78·5—81·5: F_1, head-length, 182·5—186·5 and 178·5—190·5; F_2, ditto, 172·5—182·5: G_1, head-breadth, 143·5—147·5; G_2, ditto, 134·5—146·5; H_1, head-height, 121·5—129·5; H_2, ditto, 121·5—123·5.

PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

only the four tables I to IV for inheritance in the first degree. To afford a thorough check on the results, I worked out the ratios $(A_mB_n)/\{(A_mB_n)+(A_{m+p}B_n)\}$ not for one or two pairs of columns only, but for every adjacent pair 2 : 3, 3 : 4, 4 : 6, 6 : 7. Having done this I did not think it necessary to evaluate the similar ratios for pairs of rows.

14. The results for Professor Pearson's Table I (inheritance from father to son) were most chaotic, but I persevered with the others and give the results, as regards the order assigned to the colours, in Table III. It will be seen that the orders deduced from the different comparisons in one and the same table were utterly discordant. Whatever the mathematical form of the frequency-distribution of these tables, it was clear that it could not be normal nor even a disarrangement of normally distributed frequency.

A little further inspection suggested, moreover, a certain peculiar order in the chaos: the results obtained from comparisons of the same pair of columns in different tables were much more alike than the results obtained by comparisons of different pairs of columns on the same table. Further, in 12 cases out of 16 the first place was assigned to the eye-colour of the column whose frequencies were treated as numerators, and in 13 cases out of 16 the last place fell to the eye-colour of the denominator-column.

Table III.—Result of Testing for the Order of Eye-colours, 2, 3, 4, 6 and 7, from Tables I, II, III and IV of Karl Pearson (6), 1900.

Table.	Ratio of frequencies in columns.	Order assigned.				
I. Father-son	2 : 3	2	6	7	4	3
	3 : 4	3	2	7	6	4
	4 : 6	2	4	3	7	6
	6 : 7	6	4	3	2	7
II. Father-daughter	2 : 3	6	7	2	4	3
	3 : 4	3	2	7	6	4
	4 : 6	4	2	3	7	6
	6 : 7	6	3	4	2	7
III. Mother-son	2 : 3	2	4	3	7	6
	3 : 4	3	2	6	7	4
	4 : 6	4	2	3	7	6
	6 : 7	4	3	6	2	7
IV. Mother-daughter	2 : 3	2	3	4	7	6
	3 : 4	3	2	6	7	4
	4 : 6	2	4	3	7	6
	6 : 7	6	3	4	7	2
The four tables superposed	2 : 3	2	6	7	4	3
	3 : 4	3	2	6	7	4
	4 : 6	4	2	3	7	6
	6 : 7	6	3	4	2	7

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PROCEEDINGS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PROCEEDINGS OF THE ROYAL SOCIETY A

15. When the four tables were pooled together the empirical rule thus suggested held good without exception. When the ratios of column 2 to 2+3 were taken, eye-colour 2 was placed first and 3 last. When the ratios of column 3 to 3+4 were taken, 3 was placed first and 4 last and so on (*cf.* the last division of Table III). There seemed to be only one simple explanation of these results, viz., a relative excess of frequency in the diagonal compartments 11, 22, etc., corresponding to the same eye-colour in parent and offspring, as compared with the frequencies in a normal or other isotropic distribution.

16. This is in striking contrast to the result obtained for stature and other anthropometric measurements. But, as the reader will probably remember, Mr. Galton affirms that, judging from his material,* "stature and eye-colour are not only different as qualities, but they are more contrasted in hereditary behaviour than perhaps any other common qualities. Parents of different statures usually transmit a blended heritage to their children, but parents of different eye-colours usually transmit an alternative heritage . . . . . . if one parent has a light eye-colour and the other a dark eye-colour, some of the children will, as a rule, be light and the rest dark; they will seldom be medium eye-coloured, like the children of medium eye-coloured parents." Professor Pearson, working on Mr. Galton's data, concurred, with some qualifications, in this conclusion ((6), 1900, p. 117 and p. 120). Now if the simplest possible form of such alternative inheritance hold, viz., that the offspring resemble identically (in about equal numbers) either the one parent or the other, the frequency-distribution in the contingency-table for parent and offspring must be given by a very simple rule. One-half of the offspring of parents of any one type of character must be of the same type; the other half—if we neglect the small degree of homogamy actually existing—must be distributed similarly to parents of the other sex. A like rule, *mutatis mutandis*, would hold for brother-brother or sister-sister tables.

17. Here then we seem, at first sight, to have the clue to the over-weighting of the diagonal frequencies; it may be due simply to the exclusive or alternative character of the inheritance.

When tried quantitatively, however, the theory breaks down. It gives far too great an overweighting. Thus take the figures of Table IV (condensed from Table I of Professor Pearson's Memoir, with the addition of figures for mothers from his Table III). There were 358 fathers of sons with eye-colours 1 or 2, and 269 mothers. Assuming random mating and

---

* 'Natural Inheritance,' pp. 138, 139.

PROCEEDINGS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

alternative inheritance, the number of sons with eye-colours 1 or 2 to be expected is therefore $179+\frac{269}{1000}.179 = 227{\cdot}1$, whilst only 194 were observed.

Table IV.—Eye-colour of Father.

		1, 2	3	4	5, 6	7, 8	Total.
Eye-colour of son.	1, 2	194	70	41	9	21	335
	3	83	124	41	13	23	284
	4	25	34	55	11	12	137
	5, 6	27	12	19	24	23	105
	7, 8	29	24	24	12	50	139
Total		358	264	180	69	129	1000
Mother of sons		269	289	151	144	147	1000

Table V.—Frequencies Calculated on the Assumption of Simple Alternative Inheritance.

	1, 2	3	4	5, 6	7, 8	Total.
1, 2	227·1	35·5	24·2	9·3	17·3	313·4
3	51·7	170·1	26·0	10·0	18·6	276·4
4	27·0	19·9	103·6	5·2	9·7	165·4
5, 6	25·8	19·0	13·0	39·5	9·3	106·6
7, 8	26·3	19·4	13·2	5·1	74·0	138·0
	357·9	263·9	180·0	69·1	128·9	999·8

The number to be expected with eye-colour 3 from the same fathers is $\frac{289}{1000}.179 = 51{\cdot}7$ against 83 actually observed. Proceeding in this way we obtain the figures of Table V, from which the observed frequencies diverge largely. It will be seen that the calculated frequencies in the diagonal compartments, corresponding to identity of eye-colour in father and son, are in every case much too great, whilst the calculated frequencies of the diagonal-borders are too small. The two or three other cases which I tried (though less completely) at random, gave similar results, and it would seem therefore that the theory of simple alternative inheritance or "exclusive inheritance without reversion" must be rejected or modified. It gives, as pointed out by Professor Pearson, a value of the correlation-coefficient approximately coincident with that observed, but the distribution of frequency it implies is not in accordance with fact. If the inheritance of eye-colour be exclusive or alternative in some way, if it involve for instance a Mendelian separation of characters in the formation of the germ cells, the results of such process are not visible in their simplest form, but are somehow masked.

18. The anisotropy of eye-colour tables is, however, extraordinarily well marked. I have examined the 24 tables of Professor Pearson's Memoir (6, 1900) by the short 3 × 3-fold method (§ 5), taking the colours in the Groups 123—456—78, and the three eye-colour tables of his Huxley Lecture (9, 1903) which are of 3 × 3-fold form as published. Of all the 27 tables not a single one is isotropic; in 17 of the 27 the association is positive in the first and third quadrants, negative in the second and fourth. This, and not isotropy, appears to be the characteristic form; it is the distribution of signs that would be given by an excess of frequency in the diagonal compartments, though other divergences from normality may, and indeed do, occur.

19. Further, this is not the characteristic distribution of eye-colour tables alone. An extensive series of tables for inheritance of coat-colour in horses has been given by Professor Pearson and his collaborators in the 'Philosophical Transactions' and in 'Biometrika' (1, 6, 7). These I reduced to 3 × 3-fold form by extracting the arrays for "brown," "bay," and "chestnut," the three headings (out of 16) under which 90 per cent. or more of all the entries occur, and examined for isotropy. Of the 20 tables seven only are isotropic, whilst nine exhibit the alternate distribution of signs. Of the seven tables for relationships of the first degree, only one is isotropic whilst five are of the alternate form.

20. In Professor Pearson's Huxley Lecture (9, 1903), a number of tables were given for the resemblance of qualities between brothers and sisters, 21 of which, for 7 characters, were similarly examined. Alternation of signs was the form exhibited by the three tables given for each of the characters, curliness of hair, athletics and temper, and by two tables of the three for health.* Of the 21 distributions altogether, only 6 were isotropic, 13 exhibiting alternation of signs. Taking together the eye-colour tables, coat-colour tables, and the miscellaneous tables of the "Huxley Lecture," 68 distributions were examined, of which 13 only were isotropic, for the grouping used, whilst 39 exhibited alternation of sign. These tables include certainly the great majority of the tables for inheritance of unmeasured

---

* In grouping the tables in which the original classification was higher than threefold, the following were taken as the central classes:—*Health*, normally healthy; *Hair colour*, brown; *Ability*, slow, intelligent; *Handwriting*, moderate. Professor Pearson regards the *Athletics* table as unreliable for a special reason, but it appears to me to be only a marked example of the common characteristic.

characters, with a three-fold or higher classification, and based on a considerable number of observations, that have been recently published.

21. With the exception of the tables for handwriting, the distributions for all the unmeasured characters exhibit a common peculiarity, viz., a tendency to departure from isotropy due apparently to excess of frequency for identically-named characters in the two individuals. Tables for measured characters, on the other hand, exhibit no such peculiarity so far as I have investigated them by the same rough test.

Now this result seems curious. If we had to deal with two distinct modes of inheritance, we would expect a contrast between some characters and others in the case of measured variables, and between some characters and others in the case of unmeasured qualities. The contrast actually appears to lie between measured characters as a whole and unmeasured characters as a whole. The result is such as to suggest that the excess of homonymous pairs, observed only in the latter case, may be of subjective origin, arising in some way from the mode in which different observers assign names to pairs of qualities that are more or less vaguely defined.

22. As the truth of this hypothesis would gravely affect the interpretation of many recent statistics, I decided to put it to the test of experiment, and hope to be able to publish shortly a full account of the results. Here it may be briefly stated that the effect of bias and personal equation in the experimental case was of the same kind as that noted above, the tables of observers' returns exhibiting a marked excess of homonymous pairs (together with, in some cases, an excess of contrasts). A quantitative comparison, however, between the divergence of observers' returns from truth and the divergence of some of the preceding contingency-tables from normality showed that the latter was considerably the greater. The nature of this divergence demands, accordingly, further elucidation. At present it certainly cannot, owing to its magnitude, be ascribed with confidence entirely to a subjective origin, whilst on the other hand, the absence of any confirmation from statistics of measurements, and the qualitative similarity to subjective effects, compel some reserve in assuming a biological significance.

BIBLIOGRAPHY.

1. N. Blanchard. "On the Inheritance of Coat Colour in Thoroughbred Horses," 'Biometrika,' vol. 1, p. 361 (1902).
2. W. R. Macdonell. "On Criminal Anthropometry and the Identification of Criminals," 'Biometrika,' vol. 1, p. 177 (1902).
3. —— "On the Influence of Previous Vaccination in Cases of Small-pox," 'Biometrika,' vol. 1, p. 375 (1902).

4. W. R. Macdonell. "A Further Study of Statistics relating to Vaccination and Small-pox," 'Biometrika,' vol. 2, p. 135 (1903).
5. Karl Pearson. "On the Correlation of Characters not Quantitatively Measurable," 'Phil. Trans.,' A, vol. 195, p. 1 (1900).
6. —— with Alice Lee. "On the Inheritance of Characters not capable of exact Quantitative Measurement," 'Phil. Trans.,' A, vol. 195, p. 79 (1900).
7. —— "The Law of Ancestral Heredity" (with Appendices on Inheritance in Race-horses by N. Blanchard and A. Lee), 'Biometrika,' vol. 2, p. 211 (1903).
8. —— with Alice Lee. "On the Laws of Inheritance in Man," 'Biometrika,' vol. 2, p. 357 (1903).
9. —— "On the Inheritance of the Mental and Moral Characters in Man and its Comparison with the Inheritance of the Physical Characters" (Huxley Lecture), 'Anthrop. Inst. Journ.,' vol. 33 (1903), and 'Biometrika,' vol. 3, p. 131 (1904).
10. —— with A. Barrington and A. Lee. "On the Inheritance of Coat Colour in the Greyhound," 'Biometrika,' vol. 3, p. 245 (1904).
11. —— "On the Theory of Contingency and its Relation to Association and to Normal Correlation ('Drapers' Company's Research Memoirs,' Biometric Series, vol. 1, Dulau, 1904).
12. W. F. Sheppard. "On the Application of the Theory of Error to Cases of Normal Distribution and Normal Correlation," 'Phil. Trans.,' A, vol. 192, p. 101 (1898).
13. —— "On the Calculation of the Double Integral expressing Normal Correlation," 'Cambridge Phil. Trans.,' vol. 19, p. 1 (1900).
14. —— "New Tables of the Probability Integral," 'Biometrika,' vol. 2, p. 174 (1903).
15. G. U. Yule. "On the Association of Attributes in Statistics, etc.," 'Phil. Trans.,' A, vol. 194, p. 257 (1900).
16. —— "Notes on the Theory of Association of Attributes in Statistics," 'Biometrika,' vol. 2, p. 121 (1903).